A NEW CUMULANT BASED PARAMETER ESTIMATION METHOD FOR NONCAUSAL AUTOREGRESSIVE SYSTEMS

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ABSTRACT

In this paper, a new nonlinear parameter estimation method for a noncausal autoregressive (AR) system based on a new quadratic equation relating the unknown AR parameters to higher-order (≥ 3) cumulants of non-Gaussian output measurements in the presence of additive Gaussian noise. It is applicable no matter whether or not the order of the system is known in advance; it is also applicable for the case of causal AR system. Some simulation results are offered to justify that the proposed method is effective.

1. INTRODUCTION

Autoregressive (AR) system identification with only output measurements is a well-defined problem in various science and engineering areas such as spectral estimation, speech processing, seismology, sonar, radar, radio astronomy, biomedicine, image processing, vibration analysis and oceanography. Although most of existing AR parameter estimation methods assume that the unknown AR model is causal stable, there are some cases that the underlying signal generation model is noncausal, which can be found in such as astronomical signal processing, image processing and geophysical signal processing.

A known fact is that correlation based AR parameter estimation methods such as the existing AR spectral estimators are inherently phase blind and sensitive to additive noise no matter whether the signal of interest is Gaussian or not. Recently, signal processing with higher-order statistics, known as cumulants, has drawn extensive attention because cumulants can be used to extract not only the amplitude information but also the phase information of non-Gaussian signals and they are totally zero for Gaussian processes.

Various cumulant based AR parameter estimation methods have been reported in the open literature. Most of them such as [1-6] are only applicable in the case of causal stable AR model; nevertheless some cumulant based approaches have been proposed to identify a noncausal AR model, denoted $1/A(z)$. For instance, Tugnait's exhaustive search method [7] and minimum phase-allpass (MP-AP) decomposition based method [8] and Huzii's method [9] begin with the estimation of the spectrally equivalent (SE) minimum-phase system $\widehat{A}_{MP}(z)$ by correlation based AR parameter estimation methods. From the set of all AR models SE to $1/\widehat{A}_{MP}(z)$, the exhaustive search method determines the noncausal $1/A(z)$ to be the candidate whose output cumulants match the corresponding sample cumulants best. For Huzii's method and the MP-AP decomposition based method, the given non-Gaussian data $x(k)$ are processed by the filter $\widehat{A}_{MP}(z)$ to obtain an innovations process $\widehat{u}(k)$ and the desired noncausal system $1/A(z)$ is then determined from cumulants of $\widehat{u}(k)$. Tugnait [7] also proposed an optimization method by minimizing a cost function formed of the squared errors between theoretical output correlations as well as cumulants and the corresponding sample correlations as well as sample cumulants. Giannakis [10] proposed a method which converts the noncausal AR parameter estimation problem into a causal moving average (MA) parameter estimation problem.

In this paper, we propose a new parameter estimation method for a noncausal AR system $1/A(z)$ based on a new quadratic equation relating the unknown AR parameters to cumulants of data. The proposed method finds the optimal estimate $1/\widehat{A}(z)$ through an iterative numerical optimization algorithm; it is applicable no matter whether or not the order of $1/A(z)$ is known in advance; it is also applicable for the case of causal AR model.

2. A NEW CUMULANT BASED PARAMETER ESTIMATION METHOD FOR NONCAUSAL AR SYSTEMS

Assume that $x(k)$ are the given noisy output measurements generated from a noncausal stable AR model as follows:

$$\sum_{i=-p_1}^{p_2} a(i)y(k-i) = u(k) \tag{1}$$

$$x(k) = y(k) + w(k) \tag{2}$$

where $u(k)$ is a real, zero-mean, independent identically distributed (i.i.d.) non-Gaussian process with Mth-order cumulant $\gamma_M \neq 0$ and $w(k)$ is Gaussian with unknown statistics. The pth-order ($p = p_1+p_2$) noncausal

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AR system has a transfer function $H(z) = 1/A(z)$ where

$$A(z) = \sum_{k=-p_1}^{p_2} a(k)z^{-k} = A_1(z) \cdot A_2(z) \quad (3)$$

where

$$A_1(z) = a_1(-p_1)z^{p_1} + a_1(-p_1+1)z^{p_1-1} + \ldots + a_1(0) \quad (4)$$

is a p_1th-order polynomial of z with all the roots outside the unit circle (the anticausal part of $A(z)$) and

$$A_2(z) = a_2(0) + a_2(1)z^{-1} + \ldots + a_2(p_2)z^{-p_2} \quad (5)$$

is a p_2th-order polynomial of z^{-1} with all the roots inside the unit circle (the causal part of $A(z)$). Note that correlation based AR spectral estimation methods can only provide an estimate of the SE minimum-phase $A_{MP}(z)$ given by

$$A_{MP}(z) = A_1(z^{-1}) \cdot A_2(z) \quad (6)$$

except for a scale factor.

Let $C_{M,x}(k_1, k_2, \ldots, k_{M-1})$ denote the Mth-order cumulant function of the non-Gaussian stationary process $x(k)$. It can shown that

$$\sum_{i=-p_1}^{p_2} \sum_{j=-p_1}^{p_2} a(i)a(j)C_{M,x}(k-j, -i, 0, \ldots, 0) = \mathbf{a}'\mathbf{C}(k)\mathbf{a} = \gamma_M h^{M-2}(0)\delta(k) \quad (7)$$

where $h(k)$ is the impulse response of the noncausal AR system, $\delta(k)$ is the Kronecker delta function,

$$\mathbf{a} = (a(-p_1), a(-p_1+1), \ldots, a(p_2))' \quad (8)$$

and $\mathbf{C}(k)$ is a $(p+1) \times (p+1)$ matrix whose (i,j)th component is given by

$$[\mathbf{C}(k)]_{i,j} = C_{M,x}(k+p_1-j+1, p_1-i+1, 0, \ldots, 0). \quad (9)$$

Assuming that p_1' and p_2' are chosen for p_1 and p_2, respectively, the proposed method searches for the optimum $\mathbf{a}$ by minimizing a cost function of either $J = J_1$ or $J = J_2$ through an iterative numerical optimization algorithm where

$$J_1(\mathbf{a}) = \frac{\sum_{k=-K}^{K} (\mathbf{a}'\hat{\mathbf{C}}(k)\mathbf{a})^2}{(\mathbf{a}'\hat{\mathbf{C}}(0)\mathbf{a})^2} \geq 1 \quad (10)$$

and

$$J_2(\mathbf{a}) = \frac{\sum_{k=-K,k\neq 0}^{K} (\mathbf{a}'\hat{\mathbf{C}}(k)\mathbf{a})^2}{\|\mathbf{a}\|^4} \geq 0 \quad (11)$$

in which $\hat{\mathbf{C}}(k)$ is also a $(p+1) \times (p+1)$ matrix by replacing each component of $\mathbf{C}(k)$ with the associated Mth-order sample cumulant. Some worthy remarks regarding the proposed AR parameter estimation method are summarized as follows:

(R1) The proposed AR parameter estimation method is a single-step nonlinear optimization algorithm to fit the key quadratic equation given by (7) with Mth-order sample cumulants of non-Gaussian measurements such that either J_1 or J_2 is minimum. It relies on neither the SE $A_{MP}(z)$ as exhaustive search methods [7,9] and the MP-AP decomposition based method [8] nor any conversion procedure as in Giannakis' method [10]. However, the optimum solution for $\mathbf{a}$ is not resolvable to a scale factor since $J_1(\mathbf{a}) = J_1(b\mathbf{a})$ and $J_2(\mathbf{a}) = J_2(b\mathbf{a})$ for any $b \neq 0$.

(R2) When p_1 and p_2 are known in advance, the proposed method works well while the objective function J_2 is preferred to J_1 due to its less sensitivity to initial conditions for $\mathbf{a}$ by our experience.

(R3) When $p_1' \neq p_1$ and $p_2' \neq p_2$ but $p_1' + p_2' = p_1 + p_2 = p$ is known, the optimum $\hat{A}(z)$ turns out to be an estimate $\hat{A}(z) = \alpha A(z) \cdot z^{-\tau}$ where $\tau = p_1 - p_1' = p_2' - p_2$ because cumulants are blind to time delay factors. However, $\mathbf{a}'\hat{\mathbf{C}}(0)\mathbf{a} = \gamma_M[\hat{h}(0)]^{M-2}$ (the denominator of J_1) could equal zero since $\hat{h}(k) \approx (1/\alpha)h(k+\tau)$ and therefore J_2 is preferred to J_1 for this case.

(R4) When none of p_1, p_2 and p are known *a priori*, the optimum estimate $\hat{A}(z)$ with $p_1' \geq p_1$ and $p_2' \geq p_2$ turns out to be an estimate of $\alpha A(z) \cdot z^{-\tau}$ where $p_1 - p_1' \leq \tau \leq p_2' - p_2$. However, we empirically found that the proposed method associated with J_1 always provides an optimum estimate $\hat{A}(z) \cong \alpha A(z)$ for noncausal AR systems with $max|h(k)| = |h(0)|$. The reason for this is that the minimum value of J_1 for $\tau = 0$ is always smaller than that for $\tau \neq 0$ because the value of $\mathbf{a}'\hat{\mathbf{C}}(0)\mathbf{a} = \gamma_M \hat{h}(\tau)^{M-2}$ for $\tau = 0$ is larger than that for $\tau \neq 0$ in absolute value.

(R5) The proposed AR parameter estimation method is applicable for both causal and noncausal AR systems as long as $\gamma_M \neq 0$ for any $M \geq 3$ because the causal AR(p) model is nothing but a special case of noncausal AR model for $p_1 = 0$ and $p_2 = p$.

Next, let us show some simulation results to support the proposed parameter estimation method for noncausal AR systems.

3. SIMULATION EXAMPLES

In the simulation, the driving input $u(k)$ used was a zero-mean Exponentially distributed i.i.d. sequence, data $x(k)$ of length $N = 1024$ were generated for three different signal-to-noise ratios (SNR) (10, 50 and 100) with $w(k)$ being white Gaussian. Mean and standard deviation were calculated from thirty independent estimates of $\hat{A}(z)$ with $\sum_k \hat{a}^2(k) = \sum_k a^2(k) = 1$ obtained by the proposed method with cumulant order $M = 3$ and $K = 15$ in J_1 and J_2.

Example 1. $(p_1, p_2) = (3, 2)$ are known *a priori*.

$$A(z) = -0.2071z^3 + 0.3659z^2 - 0.5247z + 0.6317 - 0.3452z^{-1} + 0.1726z^{-2} \quad (12)$$

The objective function J_2 with $p_1' = p_1$ and $p_2' = p_2$ was used for this example and the simulation results together with the true fifth-order noncausal AR parameters are shown in Table 1. One can see, from this table, that estimates $\widehat{A}(z)$ approximate the true $A(z)$ well.

Example 2. $(p_1, p_2) = (2, 0)$ are unknown but $p = p_1 + p_2 = 2$ is known.

$$A(z) = 0.6271z^2 + 0.3484z + 0.6967 \quad (13)$$

The objective function J_2 with $p_1' = 0$ and $p_2' = 2$ was used for this example and the simulation results together with the true second-order anticausal AR parameters are shown in Table 2. From this table, one can see that estimates $\widehat{a}(k)$, $k = 0, 1, 2$, are quite close to the true parameters $a(k-2)$, $k = 0, 1, 2$, respectively. These simulation results are consistent with (R3).

Example 3. None of $p_1 = 2$, $p_2 = 0$ and p are known. The same $A(z)$ given by (13) was used in this example. The objective function J_1 with $p_1' = 2$ ($\geq p_1 = 2$) and $p' = 2$ ($\geq p_2 = 0$) was used and the simulation results together with the true second-order anticausal AR parameters are shown in Table 3. Note that $h(k) = 0$ for $k > 0$ and $max|h(k)| = |h(0)| = 1$ for this case. From Table 3, one can see that estimates $\widehat{a}(-2)$, $\widehat{a}(-1)$ and $\widehat{a}(0)$ are quite close to $a(-2)$, $a(-1)$ and $a(0)$, respectively, and $\widehat{a}(1)$ and $\widehat{a}(2)$ are around zero. These simulation results also justify the statements presented in (R4).

4. CONCLUSIONS

We have presented a new cumulant based parameter estimation method for noncausal AR systems based on a new quadratic equation given by (7). The proposed method is a nonlinear estimation algorithm minimizing either J_1 given by (10) or J_2 given by (11), and the unknown pth-order AR noncausal system $1/A(z)$ where $A(z)$ is given by (3) can be estimated except for a scale factor (see (R1)). When both p_1 and p_2 are known, J_2 is preferred to J_1 (see (R2)), otherwise an unknown time delay may exist in the estimated $A(z)$; when $p = p_1 + p_2$ is known but p_1 and p_2 are not known, J_2 is also preferred to J_1 (see (R3)); when none of p_1, p_2 and p are known, J_1 is preferred to J_2 with $p_1' \geq p_1$ and $p_2' \geq p_2$ (see (R4)). The proposed method is also applicable for the case of causal AR model (see (R5)). Finally, three simulation examples were provided to justify that the proposed AR parameter estimation method is effective.

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Table 1. Simulation results of Example 1

Objective function: $J = J_2$ $p_1' = p_1 = 3$, $p_2' = p_2 = 2$ True (normalized) AR parameters: $a(-3) = -0.2071$, $a(-2) = 0.3659$, $a(-1) = -0.5247$, $a(0) = 0.6317$, $a(1) = -0.3452$, $a(2) = 0.1726$			
Estimated AR parameters (mean±standard deviation)			
	SNR=100	SNR=50	SNR=10
$\hat{a}(-3)$	−0.2119±0.0170	−0.2143±0.0183	−0.2611±0.0587
$\hat{a}(-2)$	0.3632±0.0132	0.3608±0.0155	0.3131±0.0768
$\hat{a}(-1)$	−0.5239±0.0121	−0.5212±0.0142	−0.4710±0.0670
$\hat{a}(0)$	0.6330±0.0112	0.6370±0.0127	0.6910±0.0575
$\hat{a}(1)$	−0.3423±0.0174	−0.3400±0.0215	−0.2946±0.0734
$\hat{a}(2)$	0.1720±0.0149	0.1711±0.0167	0.1548±0.0326

Table 2. Simulation results of Example 2

Objective function: $J = J_2$ $p_1 = 2$, $p_2 = 0$, $p_1' = 0$, $p_2' = 2$ True (normalized) AR parameters: $a(-2) = 0.6271$, $a(-1) = 0.3484$, $a(0) = 0.6967$,			
Estimated AR parameters (mean±standard deviation)			
	SNR=100	SNR=50	SNR=10
$\hat{a}(0)$	0.6311±0.0147	0.6307±0.0165	0.6270±0.0252
$\hat{a}(1)$	0.3609±0.0410	0.3634±0.0458	0.3647±0.0527
$\hat{a}(2)$	0.6849±0.0227	0.6836±0.0250	0.6854±0.0266

Table 3. Simulation results of Example 3

Objective function: $J = J_1$ $p_1 = 2$, $p_2 = 0$, $p_1' = 2$, $p_2' = 2$ True (normalized) AR parameters: $a(-2) = 0.6271$, $a(-1) = 0.3484$, $a(0) = 0.6967$, $a(1) = 0$, $a(2) = 0$			
Estimated AR parameters (mean±standard deviation)			
	SNR=100	SNR=50	SNR=10
$\hat{a}(-2)$	0.6311±0.0287	0.6346±0.0325	0.6546±0.0504
$\hat{a}(-1)$	0.3339±0.0391	0.3282±0.0440	0.2500±0.1092
$\hat{a}(0)$	0.6930±0.0313	0.6898±0.0386	0.6304±0.1537
$\hat{a}(1)$	−0.0134±0.0565	−0.0233±0.0660	−0.1422±0.1652
$\hat{a}(2)$	−0.0022±0.0567	−0.0050±0.0656	−0.0546±0.1297